Solving Ordinary Differential Equations in MATLAB

Fundamental Engineering Skills Workshops

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Introduction

What is a differential equation?

$$f(x) = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right)$$

Independent variable

Dependent variable

Derivatives of the dependent variable

Introduction

What is an ordinary differential equation?

"In mathematics, an ordinary differential equation or ODE is an equation containing a function of one independent variable and its derivatives." Wikipedia

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0$$

ODE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

PDE

Introduction

A simple example equation

The Problem

$$\frac{dy}{dt} = ky$$

$$y(0) = y_0$$

The Solution

$$y(t) = y_0 e^{kt}$$

Separable Equation

MATLAB

Motivation for using numerical solvers

The previous example was easy to solve by hand. In most cases though, it's not so simple.

MATLAB®

MATLAB

ODE45 - “The” MATLAB numerical solver

Runge-Kutta Method


*

- solves first order systems of ODEs
- 4th or 5th order accurate
- adaptive step sizing

Syntax:

```
[t,y] = ode45('myode',tspan,y0)
```

*Wikipedia

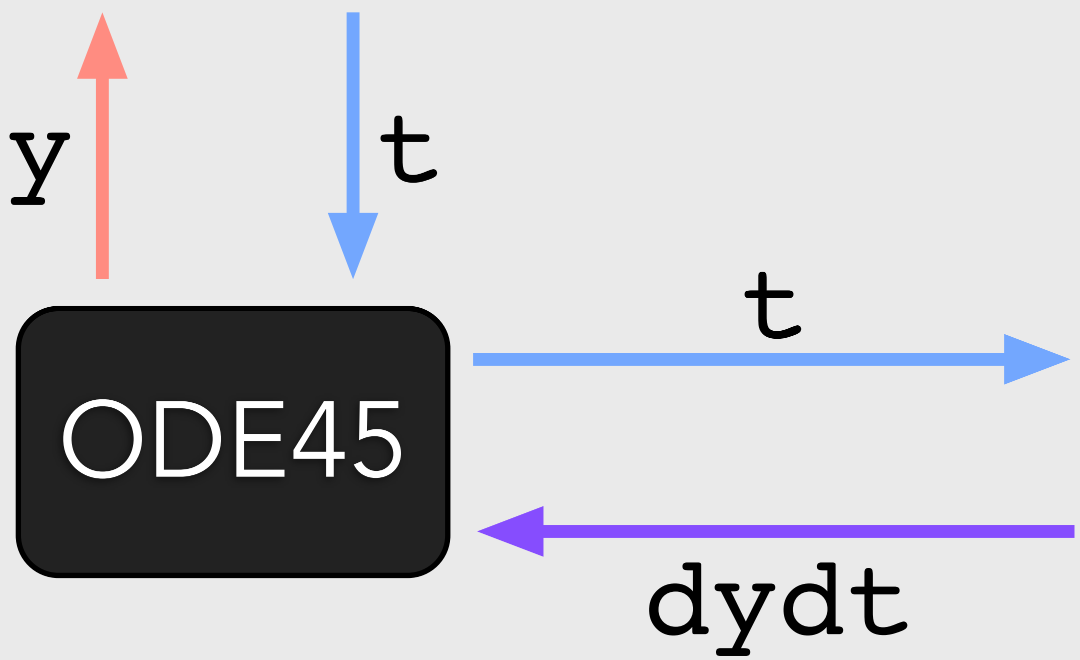
MATLAB

ODE45 - “The” MATLAB numerical solver

The Main Program

```
% BACTERIAL GROWTH
y0    = 5; %[bacteria]
tspan = [0 10]; %[hr]
[t,y] = ode45('simpleode',tspan,y0);
```

$y(0) = y_0$



The Differential Equation

```
function dydt = simpleode(t,y)
    k = 20; %[/hr]
    dydt = k*y; %[bacteria/hr]
end
```

$$\frac{dy}{dt} = ky$$

Example Problem #1

A simple example, revisited

Exponential growth

Example #1

A simple example equation, revisited

The Problem

$$\frac{dy}{dt} = ky$$

$$y(0) = y_0$$

The Solution

$$y(t) = y_0 e^{kt}$$

Separable Equation

Example Problem #2

Solving a coupled system of ODEs

Predator-Prey model

Example #2

Predator-Prey model

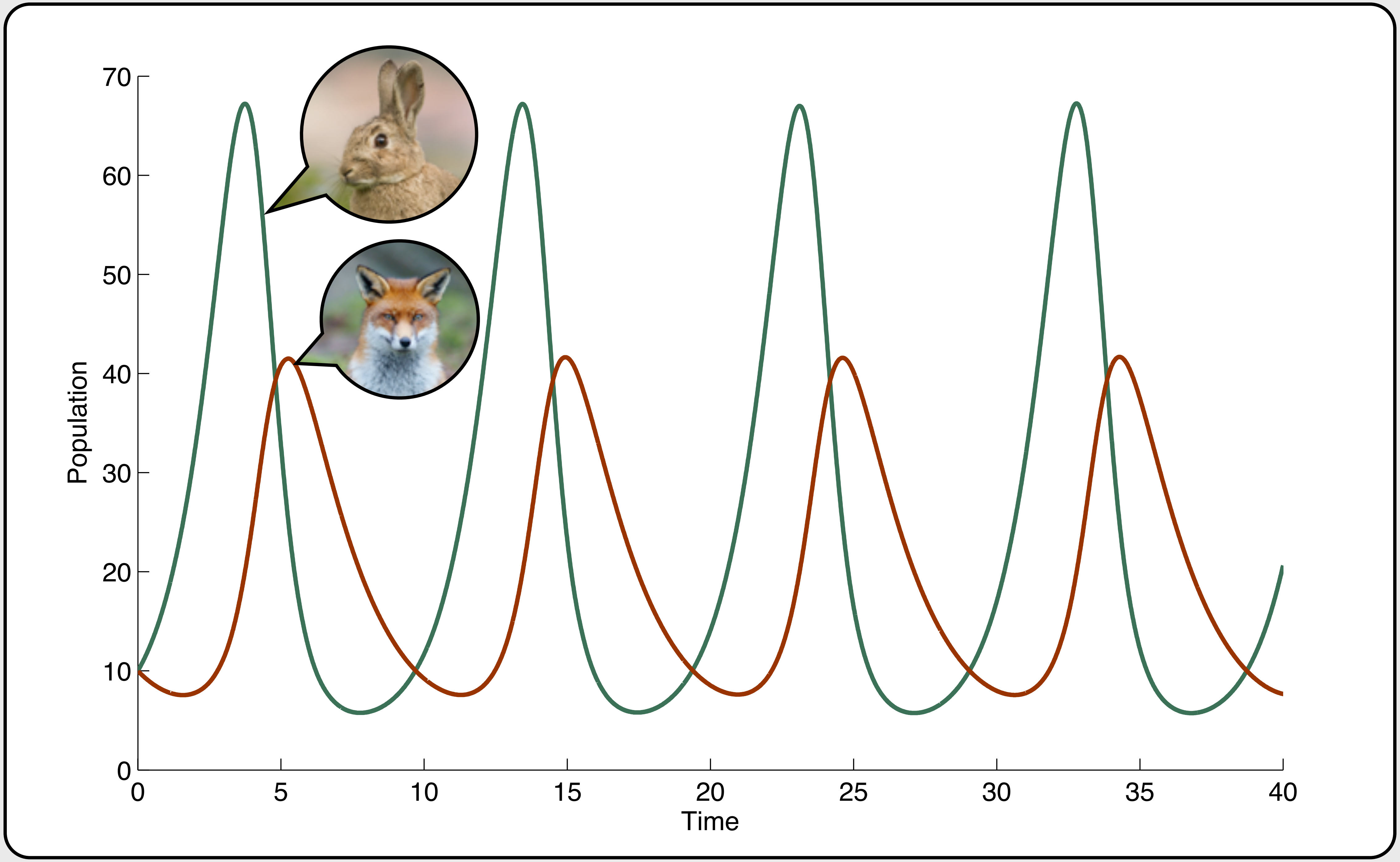
*Peter Trimming

VS

*JJ Harrison

Example #2

Predator-Prey model

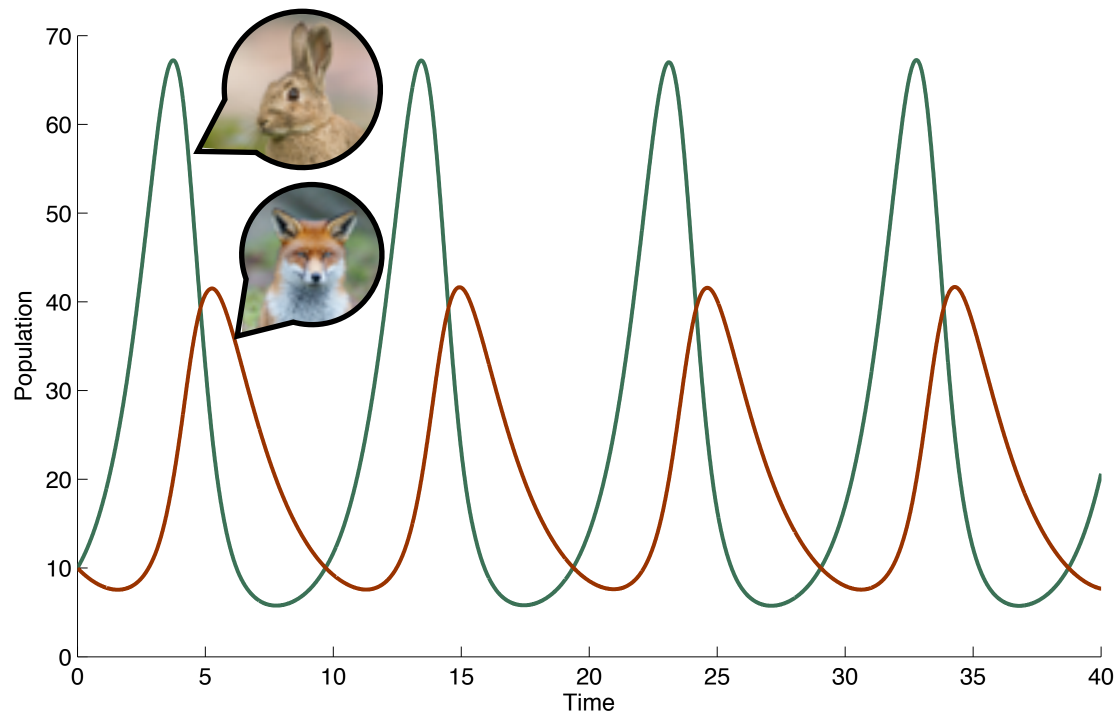


Example #2

Predator-Prey (Lotka-Volterra) model

$$\frac{dx}{dt} = (b - py)x$$

$$\frac{dy}{dt} = (rx - d)y$$



x = prey
y = predator

b = prey growth rate
p = predation rate
r = predator growth rate
d = predator death rate

Example #2

Predator-Prey (Lotka-Volterra) model

$$\text{Let } \mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}. \text{ Then } \mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (b - py)x \\ (rx - d)y \end{bmatrix}.$$

In our ODE function file...

```
% Define the two coupled equations
dxdt = (b - p*y)*x;
dydt = (r*x - d)*y;

% Define the system of equations
dppdt(1) = dxdt;
dppdt(2) = dydt;
```

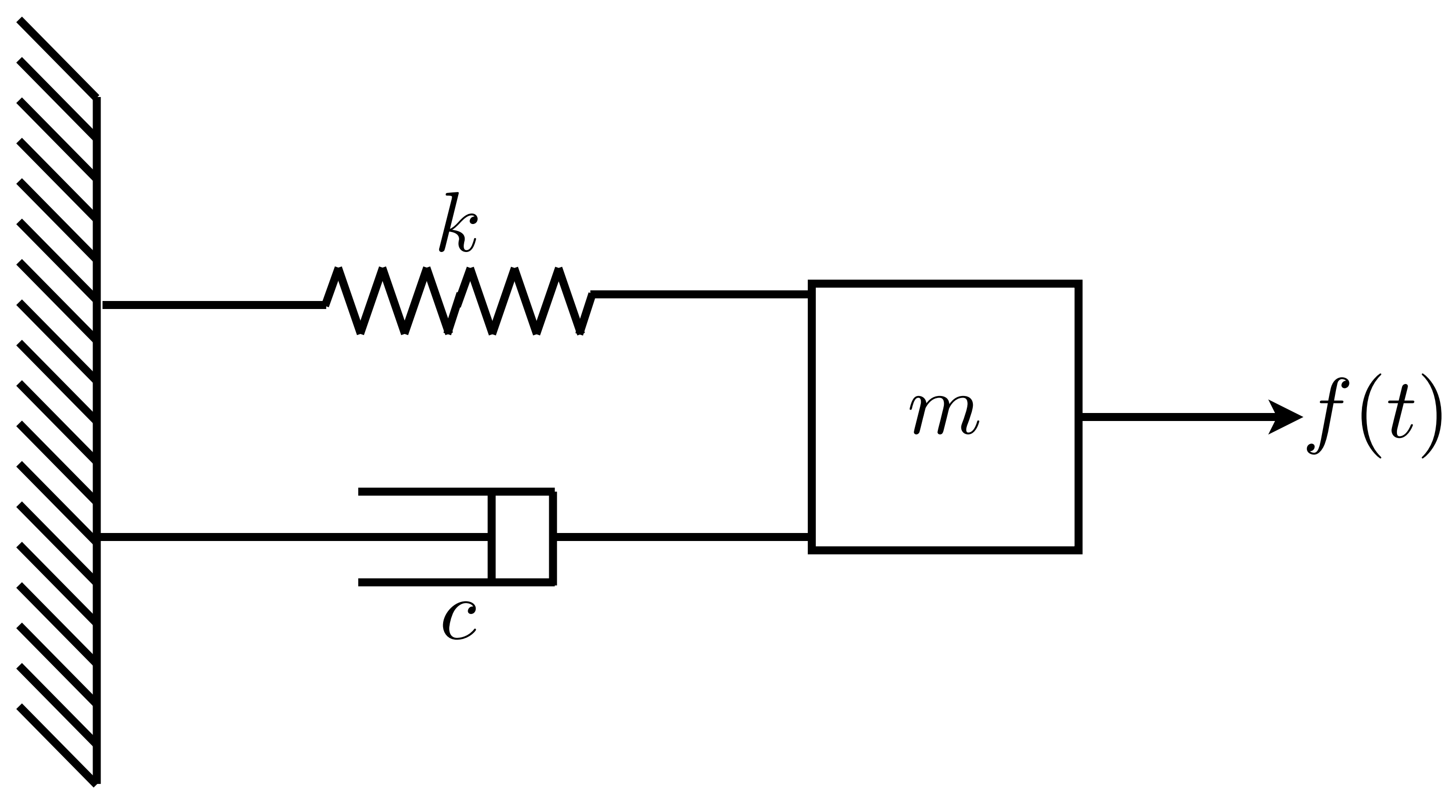
Example Problem #3

Solving a second order ODE

Spring-mass-damper system

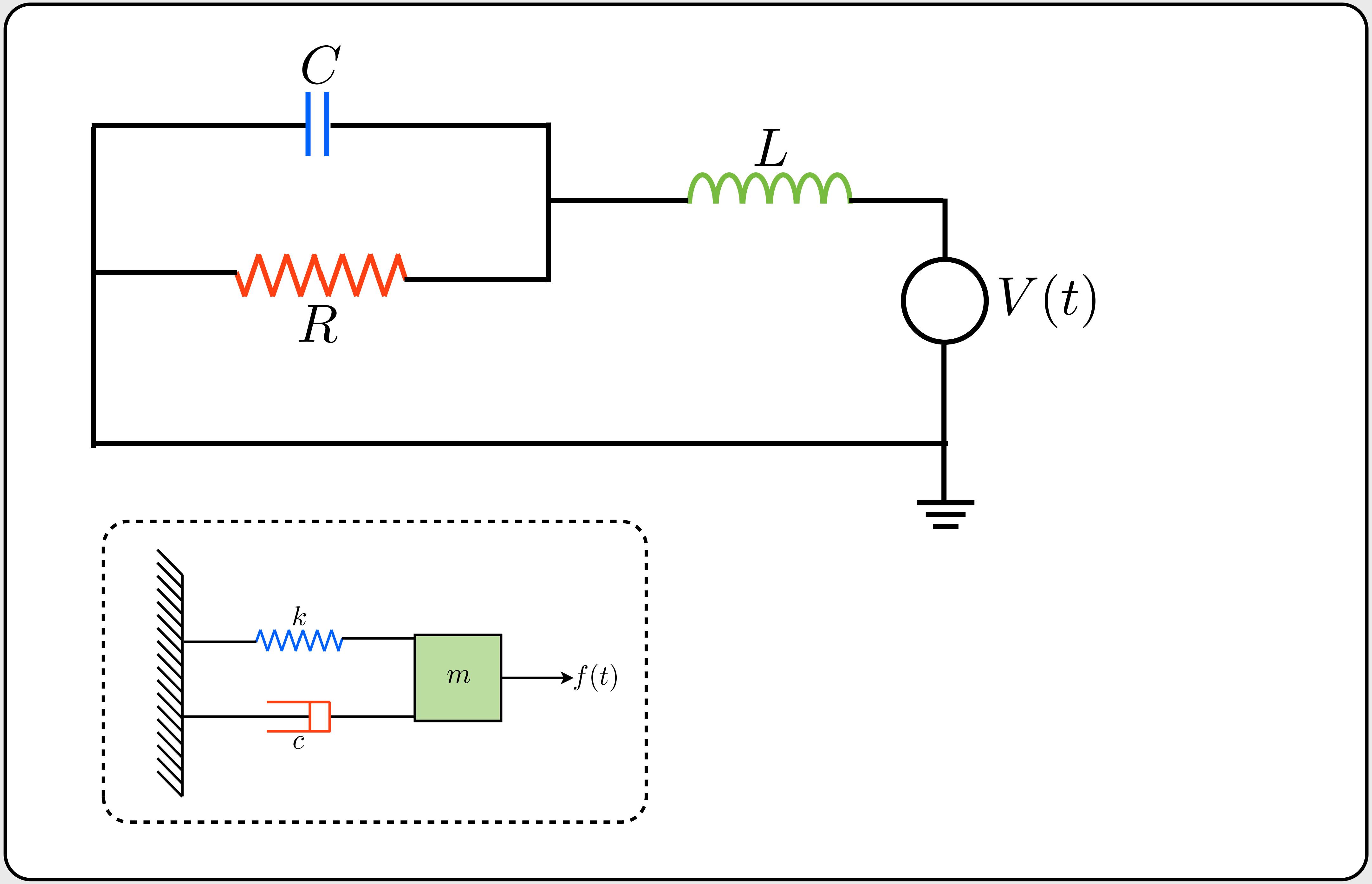
Example #3

Spring-mass-damper system



Example #3

Capacitor-inductor-resistor system



Example #3

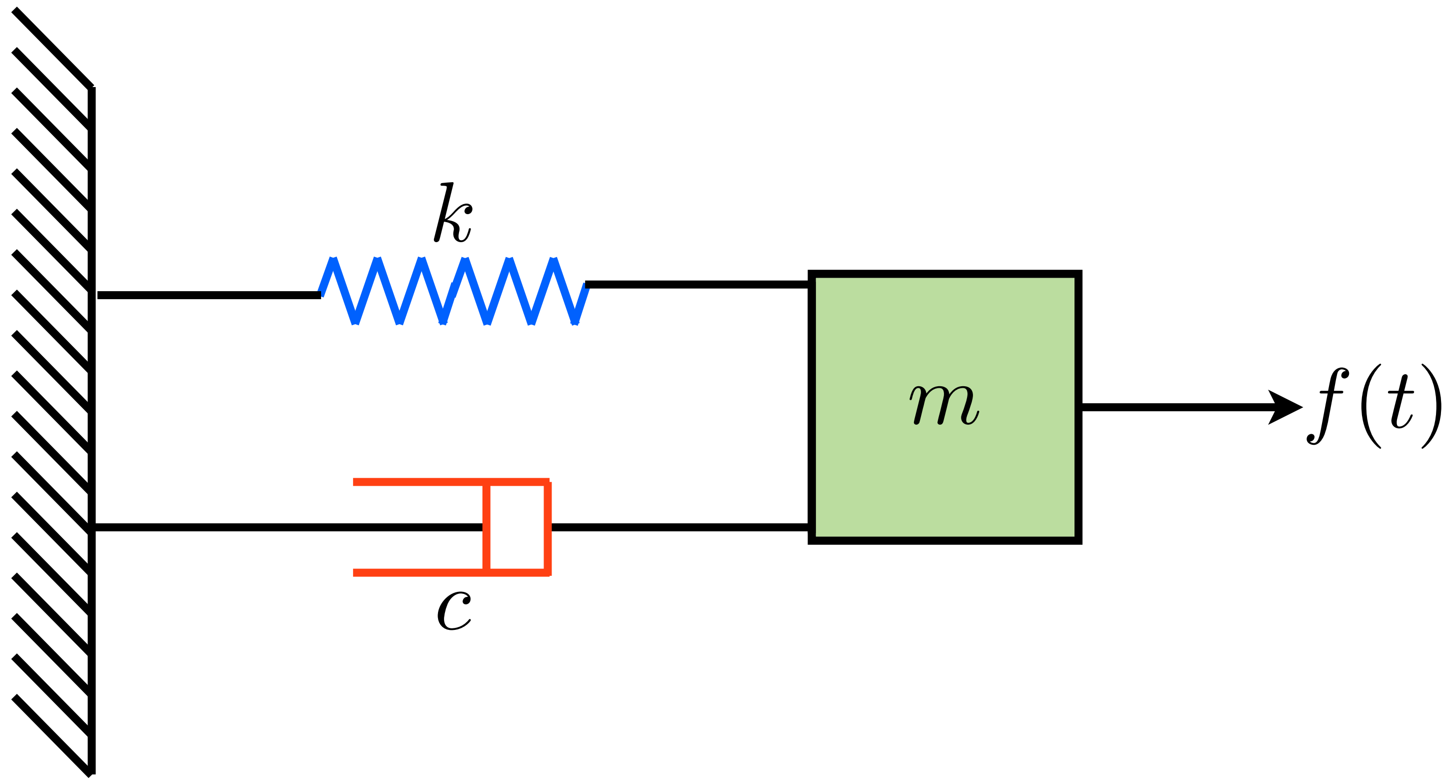
Spring-mass-damper system

$$\Sigma F_{mass} = m\ddot{x}$$

$$F_{spring} = kx$$

$$F_{damper} = c\dot{x}$$

$$F_{mass} = m\ddot{x}$$



Example #3

Spring-mass-damper system

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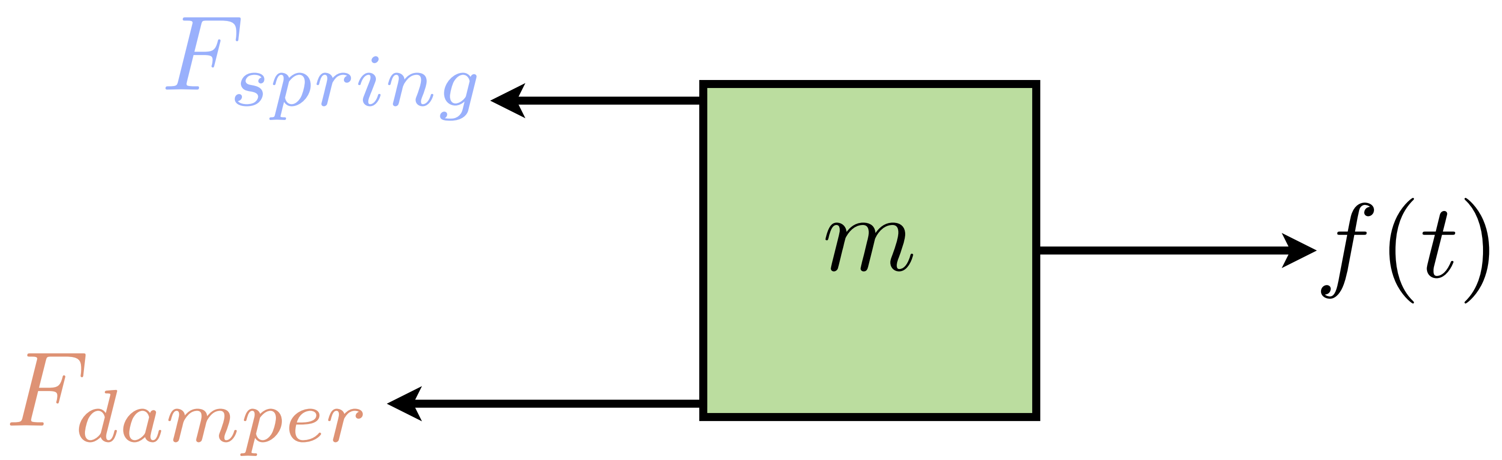
FBD



Example #3

Spring-mass-damper system

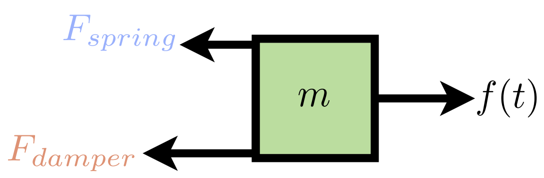
$$\Sigma F_{mass} = m\ddot{x}$$

$$F_{spring} = kx$$

$$F_{damper} = c\dot{x}$$

$$F_{mass} = m\ddot{x}$$

$$m\ddot{x} = -kx - c\dot{x} + f(t)$$



$$\ddot{x} = -\frac{k}{m}x - \frac{c}{m}\dot{x} + \frac{1}{m}f(t)$$

Example #3

Spring-mass-damper system

The Workaround

Let $y = \dot{x}$. Then $\dot{y} = \ddot{x}$.

Substitution into the ODE yields : $\dot{y} = -\frac{k}{m}x - \frac{c}{m}y + \frac{1}{m}f(t)$

Now, let $g = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\dot{g} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{k}{m}x - \frac{c}{m}y + \frac{1}{m}f(t) \end{pmatrix}$.

Example #3

Spring-mass-damper system

Now our second order equation is a system of first order equations: `ode45` will work!

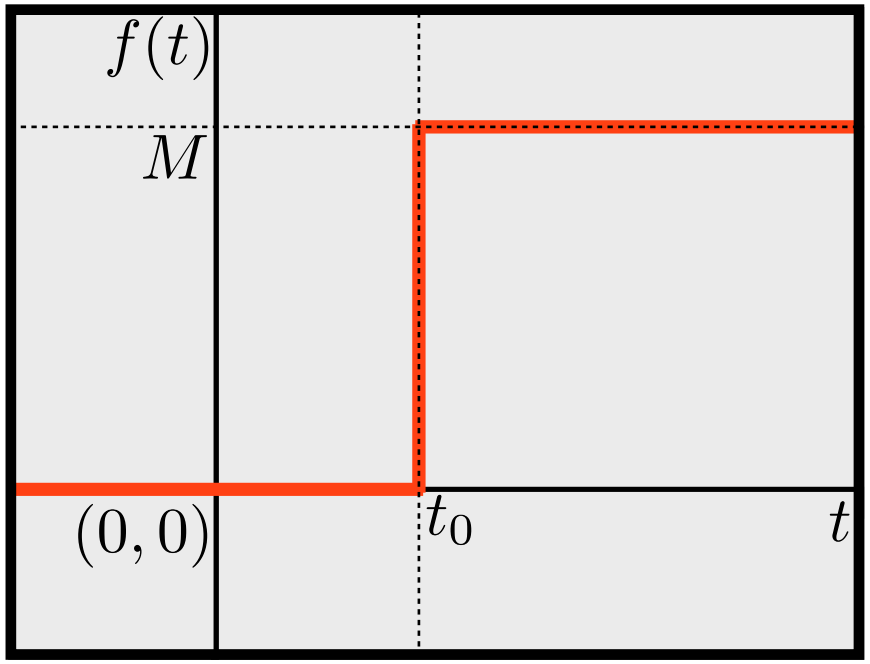
$$\dot{g} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{k}{m}x - \frac{c}{m}y + \frac{1}{m}f(t) \end{pmatrix}$$

Exercise

Code your own spring-mass-damper solver

$$\dot{g} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{k}{m}x - \frac{c}{m}y + \frac{1}{m}f(t) \end{pmatrix}$$

```
function f = force(t)
% force defines a Heaviside step
% force input at time t = t0
% with magnitude M
   t0  = 5; %[s]
   M   = 2; %[N]
   f = M*double(t >= t0);
end
```



Questions?

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